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GRONWALL – BELLMAN TYPE RETARDED INTEGRAL INEQUALITIES IN TWO VARIABLES

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ABSTRACT

In this paper we establish some new retarded integral inequalities in two variables which provide explicit bound on unknown functions. Some applications are also given to illustrate the usefulness of our results.

KEYWORDS: Explicit Bounds, Integral Inequalities, Retarded

INTRODUCTION

Integral inequalities play an important role in the qualitative analysis of the solutions of differential and integral equations; see [8,13].

In [2] Chen et al. obtained a useful upper bound on the following inequality

$$u^p(x, y) \le c + \int_{\nu(x_0)}^{\nu(x)} \int_{\delta(y_0)}^{\delta(y)} b(s, t)\,\varphi[u(s, t)]\,dt\,ds \tag{1.1}$$

And its variants under some suitable conditions on the functions involved in (1.1). In fact the result in [2] are the generalizations of the inequalities in [1]. In literature many retarded inequalities have been discovered [3-7, 9-12, 14, 15].

The main purpose of the paper is to establish explicit bounds on the general version of (1.1) which can be used more effectively in the study of certain classes of retarded differential and integral equations. Some applications are also given.

2. MAIN RESULTS

Throughout this paper, x_0, $y_0 \in R$ are fixed numbers. Let $I = [x_0, X)$ C R, $J = [y_0, Y)$ C R, and $\Delta = I \times J$ C R^2, here, we allow X or Y to be ∞. We denote by C^1 (U, V) the set of all i-times continuously differential functions of U into V, and C_0 (U, V) = C(U, V). Partial derivatives of Z are denoted by D_1Z, D_2Z, $D_{12}Z$ and so forth. Let $R_+ = [0, \infty)$.

Theorem 2.1: Let $f(x, y, s, t) \in C(\Delta \square \Delta, R_+)$ be non-decreasing in x and y for $(s, t) \in \Delta$ with $D_1 f(x, y, s, t) \in C(\Delta \times \Delta, R_+)$. Let $c \ge 0$ be a constant. If $\alpha \in C^1(I, I)$, $\beta \in C^1(J, J)$ are non-decreasing functions with $\alpha(x) \le x$ on I, $\beta(y) \le y$ on J. Moreover ϕ, $\omega \in c(R_+, R_+)$ such that ω is non-decreasing with $w(r) > 0$ for $r > 0$ and ϕ is strictly increasing with $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$. If $u \in (\Delta, R_+)$ satisfies

$$\phi(u(x,y)) \le c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x,y,s,t)\,\omega(u(s,t))\,dt\,ds \tag{2.1}$$

for all (x, y) $\in$ Δ then

$$u(x,y) \le \phi^{-1}\left[H^{-1}(H(c)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x,y,s,t)\,\omega\,dt\,ds\right]$$

For all (x, y) $\in$ $[x_0, x_1] \times [y_0, y_1]$ (2.2)

Where

$$H(r) = \int_1^r \frac{ds}{\omega(\phi^{-1}(s))}$$

Proof: Let us assume that c > 0.

Let z(x, y) be the right-hand side of inequality (2.1)

then $z(x_0, y) = z(x, y_0) = c$, $u(x, y) \le \phi^{-1}(z(x, y)$ (2.3)

Our assumption on α, β, f, ω, u imply that z is positive function that is non-decreasing in each variables.

We have

$$D_1 z(x,y) = \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} f(x,y,\alpha(x),t)\,\omega(u(\alpha(x),t))\,dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} D_1 f(x,y,s,t)\,\omega(u(s,t)\,dt\,ds$$

$$D_1 z(x,y) \le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} f(x,y,\alpha(x),t)\ \omega(\phi^{-1}(z(\alpha(x),t)))\,dt + \int_{\beta(x_0)}^{\beta(x)} \int_{\beta(y_0)}^{\beta(y)} D_1 f(x,y,s,t)\,\omega(\phi^{-1}(z(s,t))dt\,ds$$

$$D_1 z(x,y) \le \omega(\phi^{-1}(z(x,y)))\left[\int_{\beta(y_0)}^{\beta(y)} f(x,y,\alpha(x),t)\,dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} D_1 f(x,y,s,t)\,dt\,ds\right]$$

$$\frac{D_1 z(x,y)}{\omega(\phi^{-1}(z(x,y)))} \le \int_{\beta(y_0)}^{\beta(y)} f(x,y,\alpha(x),t)\,dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} D_1 f(x,y,s,t)\,dt$$

$$\text{i.e. } \frac{D_1 z(x,y)}{\omega(\phi^{-1}(z(x,y)))} \le \frac{\partial}{\partial x}\left[\int_{\alpha(x_0)}^{\alpha(x)} f(x,y,s,t)\,dt\,ds\right] \tag{2.4}$$

Integrating (2.4) with respect to x from x_0 to x and using definition of H, we get

$$H(z(x,y)) - H(z(x_0,y)) \le \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x,y,s,t)\,dt\,ds$$

$$H(z(x,y)) \le H(c) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x,y,s,t)\,dt\,ds$$

$$\therefore\ z(x,y) \le H^{-1}\left[(H(c)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x,y,s,t)\,dt\,ds\right] \tag{2.5}$$

Using inequality (2.3) in (2.5), we get

$$u(x,y) \le \phi^{-1}\left[H^{-1}(c) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x,y,s,t)\,dt\,ds\right]$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$

Theorem 2.2: Let $f(x, y, s, t) \in C(\Delta \times \Delta, R_+)$ be non-decreasing in x and y for $(s, t) \in \Delta$ with $D_1 f(x, y, s, t) \in C(\Delta \,\square\, \Delta, R_+)$. Let $c > 0$ be a constant. If $\alpha \in C^1(I, I)$, $\beta \in C^1(J, J)$ are non-decreasing functions with $\alpha(x) \le x$ on I, $b(y) \le y$ on J. Moreover $\phi \in C(R_+, R_+)$ such that it is strictly increasing with $\phi(0) = 0$ and $\phi(t) \to \infty$ as $t \to \infty$. If $u \in C(\Delta, R_+)$ satisfies

$$\phi(u(x,y)) \le c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x,y,s,t)u(s,t)\,dt\,ds \tag{2.6}$$

For all $(x, y) \in \Delta$ then

$$u(x,y) \le \phi^{-1}\left[G^{-1}\left(G(c) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x,y,s,t)\,dt\,ds\right)\right] \tag{2.7}$$

for all $(x, y) \in [x_0, x_1] \times [y_0, y_1]$

Where

$$G(r) = \int_1^r \frac{ds}{\phi^{-1}(s)}$$

Proof: Let us assume that $c > 0$

Let z(x, y) be the right-hand side of inequality (2.6) then

$$z(x_0, y) = z(x, y_0) = c \text{ and } u(x, y) < \phi^{-1}(z(x, y)) \tag{2.8}$$

Clearly z(x, y) is positive function that is non-decreasing in each variables.

we have

$$D_1 z(x_1\, y) = \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} f(x,y,\alpha(x),t)\,u(\alpha(x),t)\,dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} D_1 f(x_, y,s,t)\,u(s,t)\,dtds \tag{2.9}$$

Using (2.8) in (2.9), we have

$$D_1 z(x_1 y) = \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} f(x, y, \alpha(x), t)\, \phi^{-1}(z(\alpha(x), t))\, dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} D_1 f(x_, y, s, t)\, \phi^{-1}(z(s,t))\, dt\, ds$$

$$D_1 z(x, y) = \phi^{-1}(z(x, y)) \left[\int_{\beta(y_0)}^{\beta(y)} f(x, y, \alpha(x), t)\, dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} D_1 f(x_, y, s, t)\, dt\, ds \right]$$

$$\frac{D_1 z(x, y)}{\phi^{-1}(z(x, y))} \le \int_{\beta(y_0)}^{\beta(y)} f(x, y, \alpha(x), t)\, dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} D_1 f(x_, y, s, t)\, dt\, ds$$

i.e. $$\frac{D_1 z(x, y)}{\phi^{-1}(z(x, y))} \le \frac{\partial}{\partial x} \left[\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x_, y, s, t)\, dt\, ds \right]$$

Integrating with respect to x from x_0 to x_1 and using definition of G, we get

$$G(z(x, y)) - G(z(x_0 y)) \le \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x, y, s, t)\, dt\, ds$$

i.e. $$G(z(x, y)) \le G(c) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x, y, s, t)\, dt\, ds$$

$$z(x, y) \le G^{-1}\left[G(c) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x, y, s, t)\, dt\, ds \right] \tag{2.10}$$

Using inequality (2.8) in (2.10), we get the desired inequality (2.7).

Corollary 2.3: Let f(x, y, s, t) $\in$ C (Δ □ Δ, R_+) be non-decreasing in x and y for (s, t) $\in \Delta$ with D_1 f (x, y, s, t)$\in$ C(Δ □ Δ, R_+). Let c $\geq$ 0 be a constant. If $\alpha \in C^1$ (I, I), $\beta \in C^1$ (J, J) are non-decreasing functions with $\alpha(x) < x$ on I, β (y) $\leq$ y on J. Moreover $\phi \in$ C (R_+, R_+) which is strictly increasing with ϕ (0) = 0 and ϕ (t) $\to \infty$ as $t \to \infty$. If u$\in$ C (Δ, R_+) satisfies

$$\phi(u(x, y)) \le c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x, y, s, t)\, \phi(u(s, t))\, dt\, ds \tag{2.11}$$

For all (x, y) $\in \Delta$ then

$$u(x, y) \le \phi^{-1}\left(c\, e^{A(x, y)}\right) \tag{2.12}$$

For all x, y, $\in \Delta$ where

$$A(x, y) = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f(x, y, s, t)\, dt\, ds$$

Proof: Let $\omega = \phi$ and H (r) = l_nr in Theorem 2.1 then the corollary follows immediately from Theorem 2.1.

Some Applications

In the section, we use the results obtained in section 2 to study certain properties of boundary value problem (BVP):

$$[\phi(u(x,y))]_{xy} = F(x,y,s,t,u(\alpha(x),\beta(y))) \tag{3.1}$$

$$u(x, y_0) = e_1(x),\ u(x_0, y) = e_2(y),\ e_1(x_0) = e_2(y_0) = 0 \tag{3.2}$$

Where ϕ is defined as in Theorem 2.1,

$F \in C(\Delta \times \Delta \times R, R)$, $e_1 \in C(I, R)$, $e_2 \in C(J, R)$ are given.

Our first result deals with the boundedness of solutions.

Theorem 3.1: Consider BVP (3.1) – (3.2). If

$$|F(x, y, s, t)| \leq f(x, y, s, t)\,|u| \tag{3.3}$$

Where $f \in C(\Delta \cdot \Delta \cdot R_+)$ and

$$|\phi(e_1(x)) + \phi(e_2(y))| \leq k \text{ for some } k \geq 0 \tag{3.4}$$

Then all solutions to problem (3.1) – (3.2) satisfy

$$u(x, y) \leq \phi^{-1}\left[G^{-1}\left(G(x) + W(x,y)\right)\right]$$

For all $(x, y) \in \Delta$

Where G, G^{-1} are defined as in Theorem 2.2 if $W(x, y) = \int_{x_0}^{x}\int_{y_0}^{y} f(x,y,s,t)\,dt\,ds$ is bounded on Δ, then every solution of (3.1) – (3.2) is bounded on Δ.

Proof: If u (x, y) is the solution of the problem (3.1) – (3.2) then it satisfies the integral equation

$$\phi(u(x,y)) \leq \phi(e_1(x)) + \phi(e_2(y)) + \int_{x_0}^{x}\int_{y_0}^{y} F(x,y,\sigma,\lambda,u(\alpha(\sigma),\beta(\lambda)))\,d\sigma\,d\lambda$$

Using (3.3) and (3.4) in above inequality, we get

$$\leq k + \int_{x_0}^{x}\int_{y_0}^{y} f(x,y,\sigma,\lambda)\,|u(\alpha(\sigma),\beta(\lambda))|\,d\sigma\,d\lambda \tag{3.5}$$

Changing variables by putting $s = \alpha(\sigma)$, $t = \beta(\lambda)$,

we deduce

$$\left|\phi\left(u\left(x,y\right)\right)\right| \le k + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f\left(x,y,\alpha^{-1}\left(s\right),\beta^{-1}\left(t\right)\right)\left|u\left(s,t\right)\right| (\alpha^{-1})'(s)(\beta^{-1})'(t)\,dt\,ds \tag{3.6}$$

using inequality in Theorem (2.2), we get

$$u\left(x,y\right) \le \phi^{-1}\left[G^{-1}\left(G\left(k\right) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} f\left(x,y,\alpha^{-1}(s),\beta^{-1}(t)\right)\left(\alpha^{-1}\right)'(s)\left(\beta^{-1}\right)'\left(t\right)dt\,ds\right)\right]$$

$$u\left(x,y\right) \le \phi^{-1}\left[G^{-1}\left(G\left(k\right) + w\left(x,y\right)\right)\right]$$

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